DSF − 29/97
UAHEP9710 hep − th/9706190

2+1 Einstein Gravity as a Deformed Chern-Simons Theory

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February 1, 2008

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Abstract

The usual description of $2+1$ dimensional Einstein gravity as a Chern-Simons (CS) theory is extended to a one parameter family of descriptions of $2+1$ Einstein gravity. This is done by replacing the Poincaré gauge group symmetry by a $q-$deformed Poincaré gauge group symmetry, with the former recovered when $q \rightarrow 1$. As a result, we obtain a one parameter family of Hamiltonian formulations for $2+1$ gravity. Although formulated in terms of noncommuting dreibeins and spin-connection fields, our expression for the action and our field equations, appropriately ordered, are identical in form to the ordinary ones. Moreover, starting with a properly defined metric tensor, the usual metric theory can be built; the Christoffel symbols and space-time curvature having the usual expressions in terms of the metric tensor, and being represented by $c-$numbers. In this article, we also couple the theory to particle sources, and find that these sources carry *exotic* angular momentum. Finally, problems related to the introduction of a cosmological constant are discussed.

Introduction

In $2+1$ dimensions, as shown by E. Witten [1] (also see [2]), general relativity is equivalent to a Poincaré group gauge theory using a pure Chern-Simons (CS) action. The CS construction has among the following features: It is possible only in three dimensions, due in part, to the nondegenerate scalar product on the Poincaré algebra, which exists only in three dimensions. On shell, the gauge symmetries contain diffeomorphisms of the space-time manifold M. From the CS action one can easily read off the form of the Poisson brackets and use them to show that the constraints, which state that the field strengths vanish on a time slice of M, generate gauge transformations. The physical phase space of the theory is the space of flat connections, modulo gauge transformations. The theory is topological in nature, the only observables being global gauge invariant quantities, such as holonomies around non-contractible loops of the space manifold, and edge states if boundaries are present. [3]

In this article we show that the equivalence between general relativity in $2+1$ dimensions and CS theory persists if we replace the $2+1$ Poincaré group by a $2+1$ *quantum* Poincaré group, which we denote by $ISO_q(2,1)$, with the usual description recovered when $q \to 1$. Furthermore, we claim that all of the above mentioned features of the CS theory also persist upon making this replacement. *We thus end up with a one parameter family of descriptions of Einstein's general relativity, which exhibits a hidden quantum group gauge symmetry and a noncommutative structure.* Concerning the noncommutative structure, although we regard the space-time manifold M as a commuting manifold, i.e. it is parametrized by commuting coordinates x^μ, the fundamental fields of our theory defined on M are noncommuting. These fundamental fields are the analogues of the dreibeins and components of the spin connections of Einstein-Cartan theory. Although these fields do not commute, we can construct out of them a symmetric space-time metric, Christoffel symbols and Riemann curvature tensor, all of which commute amongst themselves. While the space-time metric does not commute with the dreibeins and the spin connections, the Christoffel symbols and the Riemann curvature tensor, being in the center of the algebra, can be thought to be ordinary numbers and hence a sensible gravity theory can be defined. In a future article[4], we shall exhibit a similar hidden quantum group gauge symmetry in four dimensional gravity, although it will obviously not be described by a CS theory.

In writing down the theory in $2+1$ dimensions we shall utilize the *deformed* CS action, which was constructed in ref. [5]. Three ingredients were found necessary before defining the deformed CS theory. One is the existence of a bicovariant calculus on the quantum group [6]. This allows for a gauge theory description based on quantum groups, à la Castellani [7]. Another ingredient is a technical requirement called 'minimality' [7],[5], while the final ingredient is the existence of a nondegenerate invariant scalar product, which allows us to write down the action.

All three of these ingredients are in fact present for $ISO_q(2,1)$. In ref. [8], it was shown that

a bicovariant calculus exists for certain quantum Poincaré groups in any number of dimensions. These quantum Poincaré groups have the feature that they contain the (undeformed) Lorentz group. For the case of three space-time dimensions, we shall give a heuristic construction of the bicovariant calculus on $ISO_q(2,1)$ in Section 1. This requires that we first define a consistent differential calculus on $ISO_q(2,1)$, and then show that it admits commuting left and right transformations, as well as left (or right) invariant one forms on the quantum group. From the bicovariant calculus on $ISO_q(2,1)$, we can then construct the corresponding gauge theory. This is done in Section 2, and there we show that it satisfies the minimality requirement.

In Section 3 we give the invariant scalar product, and then write down the CS action. As in the undeformed theory, the nondegenerate scalar product is unique to three dimensions. The equations of motion following from the CS action take the usual form, i.e., they state that the analogues of the Lorentz curvature $\mathcal{R}^a$ and torsion $\mathcal{T}^a$ vanish. However, unlike in Einstein-Cartan theory, $\mathcal{R}^a$ and $\mathcal{T}^a$, like the dreibeins and the components of the spin connections, are noncommuting. Nevertheless, as stated above, from these quantities we can construct mutually commuting space-time metrics $\mathtt{g}_{\mu\nu}$, Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ and Riemann curvatures $\mathtt{R}^\mu{}_{\nu\rho\sigma}$. This will be done in Section 4. There we also show that the previous equations of motion imply that, as usual, $\Gamma^\sigma_{\mu\nu}$ is symmetric in the lower indices and $\mathtt{R}^\mu{}_{\nu\rho\sigma}$ is zero (provided inverse dreibeins exist). We thus recover the Einstein equations in the absence of matter.

We introduce matter in the form of point particles in Section 5. The point particles in general carry a momentum and angular momentum, the former defined as the charge for $\mathcal{R}^a$ while the latter is a charge for $\mathcal{T}^a$. The momentum components commute amongst themselves, so in the absence of any sources for the torsion, we get the usual Einstein equation coupled to matter. On the other hand, for consistency, the angular momentum components do not commute amongst themselves, nor do they commute with the components of momentum, and we therefore conclude that we have particles with exotic spin. As is usual, from the Bianchi identities we can obtain a set of equations of motion for the particle degrees of freedom. In Section 6, we show how these equations, as well as the field equations, can be obtained from an action principle. It is written explicitly in terms of $ISO_q(2,1)$ variables.

The CS description of $2+1$ gravity, based on $ISO(2,1)$ or $ISO_q(2,1)$ variables applies only in the case of zero cosmological constant. In [1], Witten showed that $2+1$ gravity with a nonzero cosmological constant can also be described in terms of a CS theory, with the gauge group now being $SO(3,1)$ $[SO(2,2)]$ for positive [negative] values of the cosmological constant. In Section 7, we show that the CS construction can be applied to the case of a local q-deformed symmetry group $SO_q(3,1)$ or $SO_q(2,2)$. This yields to the usual expression for the Einstein Lagrangian in terms of dreibeins and spin connections when a cosmological constant is present, and to the usual equations of motion. However, we find that the contraction of $SO_q(3,1)$ and $SO_q(2,2)$ to $ISO_q(2,1)$ is singular, unless we first take the limit $q \to 1$. Thus this theory is not continuously connected to the deformed CS gravity with zero cosmological constant. Moreover, a symmetric and Lorentz invariant space-time metric cannot be constructed out of the dreibeins, making an important difference with the zero cosmological constant theory.

Concluding remarks are given in Section 8.

1 Bicovariant Differential Calculus on the Quantum Poincaré Group

In this Section we first define the relevant $2+1$ Poincaré group, $ISO_q(2,1)$. We then construct a differential calculus on the quantum group and show that it admits commuting left and right transformations, as well as left (or right) invariant one forms. The results are in agreement with [8] in the sense that they yield an equivalent braiding matrix, as we will see in the next Section, and the same quantum Lie algebra (apart from a rescaling of the generators).

The $2+1$ quantum Poincaré group will be expressed in terms of matrices $\ell = [\ell_{ab}]$ and vectors $z = [z_a]$. Roman letters in the beginning of the alphabet indicate Lorentz indices. We shall raise and lower them using the *off-diagonal* Lorentz metric tensor η_{ab}

$$\eta = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} . \tag{1.1}$$

We find it convenient to choose them to take values $a, b, c, d... = -1, 0, 1$. Then we have that $\eta_{ab} = \delta_{a+b,0}$.

Unlike for the ordinary Poincaré group, the matrix elements ℓ_{ab} and z_a are not c-numbers. Our choice shall be such that the ℓ_{ab}'s commute amongst themselves and the z_a's commute amongst themselves, but that they don't commute with each other. Instead, we take

$$z^a {\ell_c}^b = q^b \; {\ell_c}^b z^a \; , \tag{1.2}$$

where we introduce the real deformation parameter q. No further commutation relations are needed to define the universal enveloping algebra generated by ℓ_{ab} and z_a.

We note further that $\ell_{ab}{\ell_c}^b$ is in the center of the algebra generated by ${\ell_c}^d$ and z^b. From (1.2)

$$z^e \; {\ell_a}^d {\ell_c}^b \eta_{db} = q^{d+b} {\ell_a}^d {\ell_c}^b \eta_{db} \; z^e = {\ell_a}^g {\ell_c}^b \eta_{gb} \; z^e \; , \tag{1.3}$$

since $\eta_{db} = \delta_{d+b,0}$. Therefore we may set

$$\ell_{ab}{\ell_c}^b = \eta_{ac} \; , \tag{1.4}$$

and the commutation relations (1.2) which we have chosen are consistent with ${\ell_c}^d$ being a Lorentz matrix. Unlike with $\ell_{ab}{\ell_c}^b$, $\ell^{ba}{\ell_b}^c$ (where we now sum on the first index in ℓ) is not in the center of the algebra generated by ${\ell_f}^g$ and z^f except for certain values of b and c. From (1.2)

$$z^e \ell^{ba} {\ell_b}^c = q^{a+c} \ell^{ba} {\ell_b}^c z^e \tag{1.5}$$

Nevertheless, we may still set

$$\ell^{ba}\ell_b{}^c = \eta^{ac} \ , \tag{1.6}$$

as is done for the Lorentz group, since in so doing we are putting all noncentral components of $\ell^{ba}\ell_b{}^c$ equal to zero.

Also as is done for the Lorentz group, the determinant of ℓ may be set equal to one, as it is in the center of the algebra generated by $\ell_f{}^g$ and z^f. For this we need to introduce the totally antisymmetric tensor $\mathcal{E}_{abc}$, with $\mathcal{E}_{-1\,0\,1} = 1$. Now

$$\det(\ell)\ \mathcal{E}_{def} \equiv \mathcal{E}_{abc}\ell_d{}^a\ell_e{}^b\ell_f{}^c \tag{1.7}$$

commutes with z^g due to the identity

$$q^{a+b+c}\ \mathcal{E}_{abc} = \mathcal{E}_{abc} \ . \tag{1.8}$$

Hence we can set

$$\det(\ell) = 1 \ . \tag{1.9}$$

Our $2+1$ quantum Poincaré group is defined as the universal enveloping algebra generated by $\ell_c{}^d$ and z^b, subject to the constraints (1.4), (1.6) and (1.9). These constraints imply that $ISO_q(2,1)$ contains the (undeformed) Lorentz group.

We next wish to construct a differential calculus on the space spanned by ℓ_{ab} and z_a. For this we must specify the commutation relations for ℓ_{ab} and z_a with their exterior derivatives, the choice being consistent with (1.2). A natural choice is

$$\begin{aligned} dz^a\ell_c{}^b &= q^b\ \ell_c{}^b dz^a \\ z^a d\ell_c{}^b &= q^b\ d\ell_c{}^b z^a \\ dz^a \wedge d\ell_c{}^b &= -q^b\ d\ell_c{}^b \wedge dz^a \ , \end{aligned} \tag{1.10}$$

while assuming that the calculus on the space generated by ℓ_{ab} alone is the usual one on $SO(2,1)$, i.e.

$$\begin{aligned} d\ell_a{}^b\ell_c{}^d &= \ell_c{}^d d\ell_a{}^b \\ d\ell_a{}^b \wedge d\ell_c{}^d &= -d\ell_c{}^d \wedge d\ell_a{}^b \ , \end{aligned} \tag{1.11}$$

and the calculus on the space generated by z_a alone is the usual one on $\mathbf{R}^3$, i.e.

$$\begin{aligned} dz^b z^d &= z^d dz^b \\ dz^b \wedge dz^d &= -dz^d \wedge dz^b \ . \end{aligned} \tag{1.12}$$

Left and right transformations can be introduced in the usual way, i.e. as if the variables ℓ and z were standard Poincaré group matrices. For the case of infinitesimal *left* transformations, variations of $\ell_c{}^d$ and z^b take the form:

$$\delta_L\ell^{ba} = -\mathcal{E}_{dc}{}^b\tau_L^d\ell^{ca} \ , \qquad \delta_L z^b = -\mathcal{E}_{dc}{}^b\tau_L^d z^c + \rho_L^b \ , \tag{1.13}$$

where τ_L^b and ρ_L^b are infinitesimal parameters. ${\cal E}_{ab}{}^c$ are $so(2,1)$ structure constants

$$ {\cal E}_{ab}{}^c = {\cal E}_{abd}\;\eta^{cd}\;. \tag{1.14} $$

Our choice of indices gives ${\cal E}_{ab}{}^c = {\cal E}_{ab\;-c}$, and furthermore the result that ${\cal E}_{ab}{}^c$ is nonzero only for $a+b=c$. In addition, we have the usual identity

$$ {\cal E}_{ab}{}^c{\cal E}_{dec} = -\eta_{ad}\eta_{be} + \eta_{ae}\eta_{bd}\;. \tag{1.15} $$

The transformations (1.13) differ from the usual left Poincaré transformations because the infinitesimal parameters are not c-numbers. For this we notice that the commutation relations (1.2), (1.10), (1.11) and (1.12) are preserved under left transformations provided that

$$ [\rho_L^a, \ell^c{}_b]_{q^b} = 0\;, \quad [\rho_L^a, z_b] = 0\;, \tag{1.16} $$

where we use the notation $[A,B]_s = AB - sBA$, and τ_L^b commutes with $\ell_c{}^d$ and z^b. Similar commutation relations are assumed for τ_L^b and ρ_L^b with the one forms $d\ell_c{}^d$ and dz^b.

For the case of infinitesimal *right* transformations, variations of $\ell_c{}^d$ and z^b take the form:

$$ \delta_R\ell_c{}^d = \ell_c{}^f{\cal E}_{fe}{}^d\tau_R^e\;, \qquad \delta_R z_b = \ell_{ba}\rho_R^a\;, \tag{1.17} $$

where τ_R^b and ρ_R^b are infinitesimal parameters. This transformation differs from the usual right Poincaré transformations by the fact that the infinitesimal parameters are not c-numbers. For this we notice that the commutation relations (1.2), (1.10), (1.11) and (1.12) are preserved under left transformations provided that

$$ \begin{array}{ll} [\tau_R^a, \ell^{bc}] = 0\;, & [\tau_R^a, z^b]_{q^{-a}} = 0\;, \\ [\rho_R^a, \ell^{cb}]_{q^b} = 0\;, & [\rho_R^a, z^b]_{q^{-a}} = 0\;, \end{array} \tag{1.18} $$

along with similar commutation relations with the one forms $d\ell_c{}^d$ and dz^b. It can be easily checked that the infinitesimal left and right transformations (1.13) and (1.17) commute (at the second order in the variations).

To construct a bimodule it remains to write down the left (or right) invariant one forms. The left invariant forms are given by the usual expressions for the Poincaré group, i.e

$$ \omega_L^c = \frac{1}{2}{\cal E}_{ab}{}^c(\ell^{-1}d\ell)^{ab}\;, \quad e_L^c = (\ell^{-1}dz)^c\;, \tag{1.19} $$

although here the ordering in the expression for e_L^c is crucial. ¿From (1.2), $(\ell^{-1}dz)^c = \ell_b{}^c dz^b$ $= q^{-c}\;dz^b\ell_b{}^c$. It is straightforward to check that (1.19) is invariant under global left transformations (1.13). (By global we mean that $d\tau_L^b = d\rho_L^b = 0$.) Alternatively, the right invariant one forms are given by the usual expressions for the Poincaré group, i.e

$$ \omega_R^c = \frac{1}{2}{\cal E}_{ab}{}^c(d\ell\ell^{-1})^{ab}\;, \quad e_R^c = \ell^c{}_b d(\ell^{-1}z)^b\;. \tag{1.20} $$

In what follows, however, we shall work exclusively in terms of the left invariant forms (1.19). For convenience we shall drop the L subscripts and just denote the left invariant forms

by ω^a and e^a. They transform nontrivially under right transformations. Since from now on we shall deal with right transformations only, in the rest of the paper we shall omit writing the subscript R on the infinitesimal gauge parameters and just denote them by τ^a and ρ^a. We then have the following expressions for the variations of the left invariant forms (1.19) under infinitesimal right transformations (1.17):

$$\begin{aligned} \delta_R \omega^c &= d\tau^c + \mathcal{E}_{ab}{}^c \omega^a \tau^b \ , \\ \delta_R e^c &= d\rho^c + \mathcal{E}_{ab}{}^c (\omega^a \rho^b - \tau^a e^b) \ . \end{aligned} \tag{1.21}$$

¿From the commutation properties (1.18), we get

$$\begin{aligned} [\tau^a, \omega^b] = 0 \ , \quad [\tau^a, e^b]_{q^{-a}} = 0 \ , \\ [\rho^a, \omega^b]_{q^b} = 0 \ , \quad [\rho^a, e^b]_{q^{b-a}} = 0 \ . \end{aligned} \tag{1.22}$$

The left invariant forms (1.19) satisfy the Maurer-Cartan equations

$$\mathcal{R}^c = \mathcal{T}^c = 0 \ , \tag{1.23}$$

where

$$\begin{aligned} \mathcal{R}^c &= d\omega^c + \frac{1}{2} \mathcal{E}_{ab}{}^c \omega^a \wedge \omega^b \ , \\ \mathcal{T}^c &= de^c + \mathcal{E}_{ab}{}^c \omega^a \wedge e^b \ . \end{aligned} \tag{1.24}$$

$\mathcal{R}^c$ and $\mathcal{T}^c$ will denote the spin curvature and torsion, respectively.

2 $iso_q(2,1)$ Gauge Theory

¿From the bicovariant calculus of the preceding Section, we can write down a gauge theory based on the $iso_q(2,1)$ algebra. The algebra is described in terms of q-structure constants and braiding matrices. In this Section we shall identify these quantities and verify that they satisfy the necessary identities defining a minimal quantum group gauge theory. [7][5]

In defining the quantum group gauge theory, the right transformations (1.21) are to be regarded as gauge transformations on connection one forms ω^a and e^a [where now we no longer assume that they are pure gauges, as in eq. (1.19), and thus eq. (1.23) no longer applies]. Although the one forms have nonstandard commutation properties, the underlying space-time manifold M on which they are defined can be assumed to be an ordinary manifold, parametrized by coordinates.

The gauge transformations (1.21) can be expressed compactly by

$$\delta A^i = d\epsilon^i + C^i_{jk} A^j \epsilon^k \ , \tag{2.1}$$

where the roman letters starting from the middle of the alphabet are $iso_q(2,1)$ indices. We choose them to take values $i, j, k, \ell ... = -1, 0, 1, ..., 4$. We identify $\omega^a = A^a$ and $e^a = A^{a+3}$.

To make the connection with gravity, the former denote spin connections, while the latter are dreibein one forms. Similarly, we split the infinitesimal parameters ϵ^i into infinitesimal Lorentz transformations $\tau^a = \epsilon^a$ and translations $\rho^a = \epsilon^{a+3}$. C^k_{ij} denote the q-structure constants for $iso_q(2,1)$. From (1.21), we can make the following identifications:

$$C^c_{ab} = C^{c+3}_{a\ b+3} = \mathcal{E}_{ab}{}^c\ , \quad C^{c+3}_{a+3\ \ b} = q^{-b}\ \mathcal{E}_{ab}{}^c\ , \tag{2.2}$$

with all other components equal to zero. In the limit $q \to 1$, we recover the structure constants for $iso(2,1)$.

The commutation properties (1.22) can be expressed compactly by

$$\epsilon^i A^j = \Lambda^{ij}_{k\ell} A^k \epsilon^\ell\ , \tag{2.3}$$

where $\Lambda^{k\ell}_{ij}$ are the components of the braiding matrix. From (1.22) we can make the identifications

$$\Lambda^{cd}_{ab} = \delta^d_a\ \delta^c_b\ , \quad \Lambda^{c+3\ \ d}_{a\ \ b+3} = q^a\ \delta^d_a\ \delta^c_b\ , \quad \Lambda^{c\ \ d+3}_{a+3\ \ b} = q^{-b}\ \delta^d_a\ \delta^c_b\ , \quad \Lambda^{c+3\ \ d+3}_{a+3\ \ b+3} = q^{a-b}\ \delta^d_a\ \delta^c_b\ , \tag{2.4}$$

with all other components equal to zero. In the limit $q \to 1$, $\Lambda^{k\ell}_{ij} \to \delta^\ell_i \delta^k_j$.

For arbitrary q, the braiding matrix satisfies the Yang-Baxter equation

$$\Lambda^{ij}_{k\ell} \Lambda^{\ell m}_{sp} \Lambda^{ks}_{qu} = \Lambda^{jm}_{k\ell} \Lambda^{ik}_{qs} \Lambda^{s\ell}_{up}\ , \tag{2.5}$$

as well as

$$\Lambda^{k\ell}_{ij} \Lambda^{mn}_{k\ell} = \delta^m_i \delta^n_j\ . \tag{2.6}$$

We note that the q-structure constants are not in general antisymmetric in the lower two indices. Instead they satisfy

$$C^k_{ij} = -\Lambda^{rs}_{ij} C^r_{rs}\ , \tag{2.7}$$

along with the q-Jacobi equations:

$$C^r_{mi} C^n_{rj} - \Lambda^{k\ell}_{ij} C^r_{mk} C^n_{r\ell} - C^k_{ij} C^n_{mk} = 0\ , \tag{2.8}$$

as well as a couple of other useful identities

$$\begin{aligned} \Lambda^{ir}_{mk} \Lambda^{ks}_{n\ell} C^j_{rs} &= \Lambda^{ij}_{k\ell} C^k_{mn}\ , \\ \Lambda^{jq}_{rs} \Lambda^{si}_{k\ell} C^r_{ps} + \Lambda^{jq}_{pi} C^i_{k\ell} &= \Lambda^{sq}_{r\ell} \Lambda^{ir}_{pk} C^j_{is} + \Lambda^{jr}_{pk} C^q_{r\ell}\ . \end{aligned} \tag{2.9}$$

Using the braiding matrix and q-structure constants, we can construct the $iso_q(2,1)$ algebra. It is expressed in terms of generators T_i, $i = -1, 0, 1, ..., 4$, according to

$$T_i T_j - \Lambda^{k\ell}_{ij} T_k T_\ell = C^k_{ij} T_k\ . \tag{2.10}$$

We define $T_a = J_a$ and $T_{a+3} = P_a$, the former being analogous to angular momentum generators, while the latter are analogous to momentum generators. In terms of them (2.10) becomes

$$[J_a, J_b] \ = \ \mathcal{E}_{ab}{}^c J_c\ .$$

$$\begin{aligned} [J_a, P_b]_{q^a} &= \mathcal{E}_{ab}{}^c P_c \ , \\ [P_a, P_b]_{q^{a-b}} &= 0 \ . \end{aligned} \tag{2.11}$$

The $J - P$ commutation relations can also be written as

$$[P_a, J_b]_{q^{-b}} = q^{-b} \mathcal{E}_{ab}{}^c P_c \ . \tag{2.12}$$

In our basis, $\frac{1}{\sqrt{2}}(J_{-1} - J_1)$ can be regarded as the rotation generator, while $\frac{1}{\sqrt{2}}(J_{-1} + J_1)$ and J_0 are boost generators. In the limit $q \to 1$, one recovers the standard Poincaré algebra in three dimensions, where $\frac{1}{\sqrt{2}}(P_{-1} - P_1)$ can be interpreted as the energy, or time translation generator, while $\frac{1}{\sqrt{2}}(P_{-1}+P_1)$ and P_0 are space-like momenta, or spatial translation generators.

The relations (2.10), (2.5), (2.8) and (2.9) define a bicovariant calculus on the quantum group, while (2.6) is the condition for minimal deformations. The latter condition was utilized in finding a consistent Chern-Simons theory [5]. It was also needed for the closure of gauge transformations.[7] Finally, the identity (2.7) follows from (2.10) and (2.6). To see this just multiply both sides of (2.10) by Λ^{ij}_{rs}.

If we introduce the q-Lie algebra valued connections $A \equiv A^i T_i$, then infinitesimal gauge transformations (2.1) take the familiar form,

$$\delta A = d\epsilon + A\epsilon - \epsilon A \ , \tag{2.13}$$

where $\epsilon = \epsilon^i T_i$ and we have used (2.3). ¿From the commutation relations (1.2), (1.10), (1.11) and (1.12) for ℓ and z, we can compute the commutation relations for the spin connections and dreibeins using (1.19). We get

$$\begin{aligned} \omega^a \wedge \omega^b &= -\omega^b \wedge \omega^a \ , \\ e^a \wedge \omega^b &= -q^b \omega^b \wedge e^a \ , \\ e^a \wedge e^b &= -q^{b-a} e^b \wedge e^a \ . \end{aligned} \tag{2.14}$$

This can be expressed compactly by

$$A^i \wedge A^j = -\Lambda^{ij}_{k\ell} A^k \wedge A^\ell \ , \tag{2.15}$$

which is consistent with minimal gauge theories [7],[5]. Due to (2.6), no further conditions arise on $A^i \wedge A^j$ when multiplying both sides by $\Lambda^{k\ell}_{ij}$.

Due to the commutation relations (2.15), (2.10) the curvature two form

$$F = dA + A \wedge A \ , \tag{2.16}$$

is a q–Lie algebra element $F = F^i T_i$, the components F^i being given by

$$F^k = dA^k + \frac{1}{2} C^k_{ij} A^i \wedge A^j \ . \tag{2.17}$$

We can decompose the latter into the Lorentz curvatures $\mathcal{R}^a = F^a$ and torsions $\mathcal{T}^a = F^{a+3}$ given in (1.24). Infinitesimal gauge transformations of F take the familiar form,

$$\delta F = F\epsilon - \epsilon F \ . \tag{2.18}$$

We say that F transforms under the adjoint action of $ISO_q(2,1)$. In terms of $\mathcal{R}^a$ and $\mathcal{T}^a$ we get

$$\begin{aligned} \delta_R \mathcal{R}^c &= \mathcal{E}_{ab}{}^c \mathcal{R}^a \tau^b \;, \\ \delta_R \mathcal{T}^c &= \mathcal{E}_{ab}{}^c (\mathcal{R}^a \rho^b - \tau^a \mathcal{T}^b) \;. \end{aligned} \tag{2.19}$$

The commutation properties of F^i with ϵ^j are assumed to be

$$\epsilon^i F^j = \Lambda^{ij}_{k\ell} F^k \epsilon^\ell \;, \tag{2.20}$$

which can be expanded to

$$\begin{array}{cc} [\tau^a, \mathcal{R}^b] = 0 \;, & [\tau^a, \mathcal{T}^b]_{q^{-a}} = 0 \;, \\ [\rho^a, \mathcal{R}^b]_{q^b} = 0 \;, & [\rho^a, \mathcal{T}^b]_{q^{b-a}} = 0 \;. \end{array} \tag{2.21}$$

We should also specify the commutation relations of the curvature two forms F^i among themselves and with the connections. They have to be consistent with (2.15). We assume

$$A^i \wedge F^j = \Lambda^{ij}_{k\ell} F^k \wedge A^\ell \;, \tag{2.22}$$

and

$$F^i \wedge F^j = \Lambda^{ij}_{k\ell} F^k \wedge F^\ell \;, \tag{2.23}$$

which in terms of the Lorentz curvatures and torsions are

$$\begin{aligned} \omega^a \wedge \mathcal{R}^b &= \mathcal{R}^b \wedge \omega^a \;, \\ \omega^a \wedge \mathcal{T}^b &= q^{-a}\, \mathcal{T}^b \wedge \omega^a \;, \\ e^a \wedge \mathcal{R}^b &= q^b\, \mathcal{R}^b \wedge e^a \;, \\ e^a \wedge \mathcal{T}^b &= q^{b-a}\, \mathcal{T}^b \wedge e^a \;, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} \mathcal{R}^a \wedge \mathcal{R}^b &= \mathcal{R}^b \wedge \mathcal{R}^a \;, \\ \mathcal{R}^a \wedge \mathcal{T}^b &= q^{-a}\, \mathcal{T}^b \wedge \mathcal{R}^a \;, \\ \mathcal{T}^a \wedge \mathcal{T}^b &= q^{b-a}\, \mathcal{T}^b \wedge \mathcal{T}^a \;, \end{aligned} \tag{2.25}$$

respectively.

The Bianchi identities for the theory can be expressed in the usual way according to

$$dF + A \wedge F - F \wedge A = 0 \;, \tag{2.26}$$

or, in terms of the spin curvature and torsion, by

$$\begin{aligned} d\mathcal{R}^c &= \mathcal{E}_{ab}{}^c \mathcal{R}^a \wedge \omega^b \;, \\ d\mathcal{T}^c &= \mathcal{E}_{ab}{}^c (\mathcal{R}^a \wedge e^b - \omega^a \wedge \mathcal{T}^b) \;. \end{aligned} \tag{2.27}$$

Once again the ordering is crucial.

3 Deformed Chern-Simons Gravity

We now utilize [5] and write down a CS action for the quantum group gauge theory of Section 2. For this we specialize to the case where the space-time manifold M is three dimensional.

In order to write down the Chern-Simons theory, we will need to specify an invariant q-group metric g_{ij}. Invariance for minimal gauge groups means that [5]

$$g_{\ell k} C^{\ell}_{ij} = g_{i\ell} C^{\ell}_{jk} \ . \tag{3.1}$$

This condition is satisfied for the following choice of g_{ij}:

$$g_{a\ 3-a} = 1 \ , \quad g_{3-a\ a} = q^{-a} \ , \quad (\text{no sum on } a) \tag{3.2}$$

with all other components equal to zero. Here we note that the q-group metric is not symmetric. Upon using the notation of ref. [5], $g_{ij} =< T_i, T_j >$, we have that

$$\begin{aligned} < J_a, P_b >= \eta_{ab} \ , \quad < P_a, J_b >= q^a \eta_{ab} \ , \\ < J_a, J_b >=< P_a, P_b >= 0 \ . \end{aligned} \tag{3.3}$$

The Chern-Simons Lagrangian density found in [5] is given by

$$\begin{aligned} \mathcal{L}_{CS} &= < dA + \frac{2}{3} A^2, A > \\ &= g_{ij}\ (dA^i + \frac{1}{3} C^i_{k\ell} A^k \wedge A^\ell) \wedge A^j \ . \end{aligned} \tag{3.4}$$

For minimally deformed gauge theories it was shown to be gauge invariant (up to a total derivative), and to have its exterior product equal to $< F, F >$. Upon using (2.2) and (2.24) we are able to write (3.4) as follows:

$$\mathcal{L}_{CS} = 2(d\omega^c + \frac{1}{2} \mathcal{E}_{ab}{}^{c} \omega^a \wedge \omega^b) \wedge e_c \ + \ q^a d(e^a \wedge \omega_a) \ , \tag{3.5}$$

where $e_a = \eta_{ab} e^b = e^{-a}$, while its exterior derivative is given by

$$< F, F >= 2\mathcal{R}^c \wedge \mathcal{T}_c \ , \tag{3.6}$$

using (2.25). If the three manifold M has no boundary then the associated Chern-Simons action is just

$$\mathcal{S}_{CS} = \int_M \mathcal{L}_{CS} = \int_M 2\mathcal{R}^c \wedge e_c \ , \tag{3.7}$$

which is the usual expression for Chern-Simons gravity[1], although once again here the ordering is crucial. Under gauge transformations (1.21) and (2.19),

$$\delta(\mathcal{R}^c \wedge e_c) = d(\mathcal{R}_c \rho^c) \ , \tag{3.8}$$

and hence the action $\mathcal{S}_{CS}$ is gauge invariant.

The equation of motion $\mathcal{R}^c = 0$ follows immediately from extremizing the action with respect to variations in the dreibeins. In varying the spin connection, let us assume, as usual, that the one forms ω^a anticommute with their variations, i.e.

$$\delta\omega^a \wedge \omega^b = -\omega^b \wedge \delta\omega^a \ , \tag{3.9}$$

This is consistent with the variations $\delta\omega^a$ being gauge variations $\delta_R\omega^a$ as in (1.21). Then after integrating by parts,

$$\begin{aligned} \delta\mathcal{S}_{CS} &= \int_M 2\delta\omega^a \wedge (de_a + \mathcal{E}_{ab}{}^c\omega^b \wedge e_c) \\ &= \int_M 2\delta\omega^a \wedge \mathcal{T}_a \ , \end{aligned} \tag{3.10}$$

yielding $\mathcal{T}_a = 0$.

As expected, the equations of motion associated with $\mathcal{L}_{CS}$ state that the $iso_q(2,1)$ curvature vanishes, i.e. we recover eq. (1.23). On simply connected space-time manifolds M, we then get vacuum solutions of the form (1.19).

¿From ref. [5] we know that $F = 0$ follows from (3.4), provided that the matrix $H_{ij} = \Lambda^{k\ell}_{ij} g_{k\ell} + g_{ij}$ is nondegenerate, which is the case for our choice of q-group metric tensor g_{ij}. An alternative metric tensor exists which is instead degenerate, leading also to a degenerate H_{ij}. It is simply the Cartan-Killing metric for the $so(2,1)$ Lie-subalgebra:

$$g^{(CK)}_{ab} = \eta_{ab} \ , \tag{3.11}$$

with all remaining components $g^{(CK)}_{ij}$ equal to zero. It is easy to show that (3.11) satisfies the invariance condition (3.1). [More generally, any linear combination of (3.2) and (3.11) satisfies the invariance condition.] ¿From this metric tensor we get the standard $so(2,1)$ Chern-Simons Lagrangian

$$\mathcal{L}'_{CS} = (d\omega^c + \frac{1}{3}\mathcal{E}_{ab}{}^c\omega^a \wedge \omega^b) \wedge \omega_c \ , \tag{3.12}$$

whose associated equations of motion are just $\mathcal{R}^c = 0$. Thus from $\mathcal{L}'_{CS}$ we obtain no conditions on the torsion. Furthermore, the classical dynamics is preserved if we add the term (3.12) to $\mathcal{L}_{CS}$, although the resulting Poisson structure will not be.

Because the action (3.7) (and also the one obtained from $\mathcal{L}'_{CS}$) does not require a space-time metric (although it does require the Lorentz metric η_{ab} and the q-group metric g_{ij} and/or $g^{(CK)}_{ab}$) , it is, as usual, invariant under diffeomorphisms, and thus topological. Infinitesimal diffeomorphisms of the connection one form are given by

$$\delta_\xi A = \mathcal{L}_\xi A = i_\xi dA + d i_\xi A \ , \tag{3.13}$$

where ξ is an infinitesimal vector field, and $\mathcal{L}_\xi$ and i_ξ denote, respectively, the Lie derivative and contraction with ξ. Eq. (3.13) can be reexpressed in terms of the curvature two form F according to

$$\delta_\xi A = i_\xi F + d i_\xi A + A i_\xi A - i_\xi A A \ , \tag{3.14}$$

which is in the form of an infinitesimal gauge transformation after imposing the equations of motion $F=0$ and setting $\epsilon = i_\xi A$. In terms of the spin connections and dreibeins, the latter implies $\tau^a = \xi^\mu \omega^a_\mu$ and $\rho^a = \xi^\mu e^a_\mu$. The fact that the two symmetry generators coincide is the reason why there aren't two distinct sets of symmetry generators, one for the gauge symmetry and one for diffeomorphisms, in the Hamiltonian formulation of the theory[5]. This result has been noted by many authors [9] for undeformed Chern-Simons theory. The novelty here is that the components of ϵ and of A are q-numbers, although we assume, as usual, that those of ξ are ordinary c-numbers. Furthermore, from (2.3) and (2.15), the commutation properties of ϵ and $i_\xi A$ are identical.

We now turn to the canonical formalism. We thus consider the action (3.4) on a three manifold $M = \Sigma \times R$, where Σ is a two-manifold playing the role of space, while R accounts for time. In ref.[5] it was shown how to write consistent equal time Poisson brackets for the space components of the q-CS field for a generic q-deformed CS theory and we refer the reader to that paper for the details. Here we just write the result; the only nonvanishing Poisson brackets are:

$$\{e^a_i(\bar{x}), \omega^b_j(\bar{y})\} = -q^b \{\omega^b_j(\bar{y}), e^a_i(\bar{x})\} = \eta^{ab} \epsilon_{ij} \delta^2(\bar{x}-\bar{y}),$$
$$i,j = 1,2 \quad \epsilon_{12} = -\epsilon_{21} = 1 \ , \tag{3.15}$$

where $\bar{x}$ and $\bar{y}$ are points of Σ. Notice that the q-Poisson brackets are not antisymmetric: the extra factor of q^b that arises after the interchange of the arguments reflects the commutation relations (2.14) and is a consequence of the fact that the invariant metric (3.2) on the q-Poincaré algebra is not symmetric. The time components of the q-CS field play the role of Lagrange multipliers for the constraints

$$\mathcal{T}^a_{ij} \approx 0 \ , \quad \mathcal{R}^a_{ij} \approx 0 \ . \tag{3.16}$$

In ref.[5] it is shown that they are closed with respect to the Poisson brackets (3.15), and that they generate time-independent gauge transformations, namely local Poincaré transformations on Σ.

4 Recovering $2+1$ Einstein Gravity

Here we show that from the noncommuting generators of the deformed CS theory, we can define a space-time metric $\mathtt{g}_{\mu\nu}$, Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ and a Riemann curvature $\mathtt{R}^\mu{}_{\nu\rho\sigma}$, all of which mutually commute. Furthermore, from the equations of motion $F=0$ of the previous Section, we get that $\Gamma^\sigma_{\mu\nu}$ is symmetric in the lower two indices and $\mathtt{R}^\mu{}_{\nu\rho\sigma}$ vanishes, provided 'inverse' dreibeins exist. In addition, we end up with the usual expression for $\mathtt{R}^\mu{}_{\nu\rho\sigma}$ and $\Gamma^\sigma_{\mu\nu}$ in terms of $\mathtt{g}_{\mu\nu}$.

We first introduce the space-time metric on M. Its definition, however, is not unique. If we define it as is standardly done by

$$\eta_{ab}\, e^a_\mu e^b_\nu = e^{-1}_\mu e^1_\nu + e^0_\mu e^0_\nu + e^1_\mu e^{-1}_\nu \ , \tag{4.1}$$

(e^a_μ denoting the space-time components of the dreibein one form e^a ; μ and ν being space-time indices) then it, like the q-group metric, is not symmetric. To show this, we need the commutation relations between different space-time components of the connection one forms A^i. Since the space–time manifold is described by ordinary commuting coordinates we may replace (2.14) by the stronger conditions

$$\begin{aligned} \omega^a_\mu \omega^b_\nu &= \omega^b_\nu \omega^a_\mu \ , \\ e^a_\mu \omega^b_\nu &= q^b \ \omega^b_\nu e^a_\mu \ , \\ e^a_\mu e^b_\nu &= q^{b-a} \ e^b_\nu e^a_\mu \ . \end{aligned} \tag{4.2}$$

Then (4.1) is equal to

$$q^2 e^1_\nu e^{-1}_\mu + e^0_\nu e^0_\mu + q^{-2} e^{-1}_\nu e^1_\mu \ , \tag{4.3}$$

and hence it is not symmetric. Moreover (4.1) is not invariant under local Lorentz transformations, as we would like it to be.

Alternatively, a symmetric space-time metric $\mathtt{g}_{\mu\nu}$, which is also invariant under local Lorentz transformations, can be introduced on M by deforming the usual definition (4.1) to

$$\mathtt{g}_{\mu\nu} = q^{\frac{a-b}{2}} \ \eta_{ab} \ e^a_\mu e^b_\nu \ . \tag{4.4}$$

The ordering of the dreibein components is crucial. Although $\mathtt{g}_{\mu\nu}$ is symmetric, the tensor elements are not c-numbers. From (4.2), we get that different components of $\mathtt{g}_{\mu\nu}$ do not commute with the $iso_q(2,1)$ connection one forms ω^a and e^a,

$$\begin{aligned} \mathtt{g}_{\mu\nu} \ \omega^a_\rho &= q^{2a} \ \omega^a_\rho \ \mathtt{g}_{\mu\nu} \ , \\ \mathtt{g}_{\mu\nu} \ e^a_\rho &= q^{2a} \ e^a_\rho \ \mathtt{g}_{\mu\nu} \ . \end{aligned} \tag{4.5}$$

Nevertheless, components of $\mathtt{g}_{\mu\nu}$ do commute with themselves.

By substituting the vacuum solution (1.19) into the definition (4.4) of the symmetrized metric tensor we get the usual result

$$\begin{aligned} \mathtt{g}_{\mu\nu} &= q^a \ \eta_{ab} \ {\ell_d}^z \partial_\mu z^d \ {\ell_e}^b \partial_\nu z^e \\ &= q^{a+b} \ \eta_{ab} \ {\ell_d}^z {\ell_e}^b \ \partial_\mu z^d \partial_\nu a^e \\ &= \eta_{de} \partial_\mu z^d \partial_\nu z^e \ . \end{aligned} \tag{4.6}$$

(Remember though that the z's in the above equation are not c-numbers, as they do not commute with the ℓ's, see eq.(1.2).) A local Lorentz frame is then obtained by choosing the commuting functions z^a equal to the space-time coordinates of the manifold M.

Next we write the Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ in terms of the (symmetrized) space-time metric tensor. We introduce the Christoffel symbols along with the spin connections in the definition of the covariant derivative, and demand, as is usual, that the covariant derivative of the dreibein components vanish, i.e. $\mathtt{D}_\mu e^b_\nu = 0$, where

$$\mathtt{D}_\mu e^b_\nu = \mathcal{D}_\mu e^b_\nu - \Gamma^\sigma_{\mu\nu} e^b_\sigma \ , \tag{4.7}$$

and

$$\mathcal{D}_\mu e^b_\nu = \partial_\mu e^b_\nu + \mathcal{E}_{cd}{}^b \omega^c_\mu e^d_\nu \ . \tag{4.8}$$

The torsion being zero is consistent with the Christoffel symbols being symmetric in the lower two indices. Upon assuming

$$\begin{aligned} \partial_\rho e^a_\mu \omega^b_\nu &= q^b \ \omega^b_\nu \partial_\rho e^a_\mu \ , \\ \partial_\rho e^a_\mu e^b_\nu &= q^{b-a} \ e^b_\nu \ \partial_\rho e^a_\mu \ , \end{aligned} \tag{4.9}$$

along with (4.2), we deduce from (4.7) that the Christoffel symbols commute with the dreibeins and spin connections, and then consequently, also with the space-time metric and with themselves.

¿From $\mathtt{D}_\mu e^b_\nu = 0$, we can eliminate the spin connections, if we multiply on the left by $q^a \eta_{ab} e^a_\rho$, sum over the b index, and symmetrize with respect to the space-time indices ν and ρ. The result is

$$0 = q^a \eta_{ab} [e^a_\rho \partial_\mu e^b_\nu + e^a_\nu \partial_\mu e^b_\rho - e^a_\rho e^b_\sigma \Gamma^\sigma_{\mu\nu} - e^a_\nu e^b_\sigma \Gamma^\sigma_{\mu\rho}] \ . \tag{4.10}$$

Next we add to this the equation obtained by switching μ and ν, and subtract the equation obtained by replacing indices (μ, ν, ρ) by (ρ, μ, ν). We can then isolate $\Gamma^\sigma_{\mu\nu}$ according to

$$2q^a \eta_{ab} e^a_\rho e^b_\sigma \Gamma^\sigma_{\mu\nu} = q^a \eta_{ab} [e^a_\rho (\partial_\mu e^b_\nu + \partial_\nu e^b_\mu) + e^a_\nu (\partial_\mu e^b_\rho - \partial_\rho e^b_\mu) + e^a_\mu (\partial_\nu e^b_\rho - \partial_\rho e^b_\nu)] \tag{4.11}$$

or

$$2\mathtt{g}_{\rho\sigma} \Gamma^\sigma_{\mu\nu} = \partial_\mu \mathtt{g}_{\rho\nu} + \partial_\nu \mathtt{g}_{\rho\mu} - \partial_\rho \mathtt{g}_{\nu\mu} \ . \tag{4.12}$$

In order to solve the above equation for the Christoffel symbols, we need to invert the metric $\mathtt{g}_{\mu\nu}$. To do this, we start from defining the inverses e^μ_a of the co-vectors e^a_μ. All we need to do is to enlarge our algebra by a new element called $\mathtt{e}^{-1}$ fulfilling the following commutation relations:

$$\begin{aligned} \mathtt{e}^{-1} e^a_\mu &= q^{-3a} e^a_\mu \mathtt{e}^{-1} \quad , \\ \mathtt{e}^{-1} \omega^a_\mu &= q^{-3a} \omega^a_\mu \mathtt{e}^{-1} \ , \end{aligned} \tag{4.13}$$

and such that

$$\mathtt{e}^{-1} \mathtt{e} = 1 \ , \tag{4.14}$$

where $\mathtt{e}$ is the determinant:

$$\mathtt{e} = \epsilon^{\mu\nu\rho} e^{-1}_\mu e^0_\nu e^1_\rho \ , \tag{4.15}$$

and $\epsilon^{\mu\nu\rho}$ is antisymmetric in the space-time indices. It is easy to verify that eq.(4.14) is consistent, because its left hand side commutes with everything. Moreover, it is also true that $\mathtt{e}^{-1}\mathtt{e} = \mathtt{e}\mathtt{e}^{-1}$. Using $\mathtt{e}^{-1}$ we can now define the inverses of the dreibeins to be:

$$e^\mu_a \equiv -\frac{1}{2} \hat{\mathcal{E}}_{abc} \epsilon^{\mu\nu\rho} e^b_\nu e^c_\rho \mathtt{e}^{-1} \ , \tag{4.16}$$

where the q-antisymmetric tensor $\hat{\mathcal{E}}_{abc}$ is defined such that

$$\hat{\mathcal{E}}_{abc} e^a \wedge e^b \wedge e^c = e^{-1} \wedge e^0 \wedge e^1 \ , \quad \text{no sum on } a \ , b \ , c \ . \tag{4.17}$$

The solution to this equation may be expressed by $\hat{\mathcal{E}}_{abc} = q^{a-c+2}\mathcal{E}_{abc}$. It can be checked that e^μ_a satisfy the usual properties:

$$
\begin{aligned}
e^a_\mu e^\mu_b &= e^\mu_b e^a_\mu = \delta^a_b \ ; \\
e^a_\nu e^\mu_a &= e^\mu_a e^a_\nu = \delta^\mu_\nu \ .
\end{aligned}
\tag{4.18}
$$

Their commutation properties can be worked out to be:

$$
\begin{aligned}
e^\mu_a e^\nu_b &= q^{b-a} e^\nu_b e^\mu_a \ , \\
e^\mu_a e^b_\nu &= q^{a-b} e^b_\nu e^\mu_a \ , \\
e^\mu_a \omega^b_\nu &= q^{-b} \omega^b_\nu e^\mu_a \ .
\end{aligned}
\tag{4.19}
$$

Using the vectors e^μ_a we can now define the inverse of the metric $\mathsf{g}_{\mu\nu}$ as:

$$
\mathsf{g}^{\mu\nu} \equiv q^{\frac{a-b}{2}} \eta^{ab} e^\mu_a e^\nu_b \ , \tag{4.20}
$$

where η^{ab} is the ordinary inverse of the matrix η_{ab}. The inverse metric $\mathsf{g}^{\mu\nu}$ is symmetric and satisfies the standard conditions:

$$
\mathsf{g}^{\mu\rho}\mathsf{g}_{\rho\nu} = \mathsf{g}_{\nu\rho}\mathsf{g}^{\rho\mu} = \delta^\mu_\nu \ . \tag{4.21}
$$

Moreover the $\mathsf{g}^{\mu\nu}$ commute among themselves and with the $\mathsf{g}_{\rho\sigma}$ (but do not commute with the dreibeins and with the spin connection), thus in eqs.(4.12) we can solve for the Christoffel symbols without any ordering problems. The final expression of $\Gamma^\sigma_{\mu\nu}$ in terms of the metric and of its inverse coincides with the standard one.

It remains to recover the Einstein equations, which in vacuum state that the space-time curvature $\mathtt{R}^\mu{}_{\nu\rho\sigma}$ is zero. $\mathtt{R}^\mu{}_{\nu\rho\sigma}$ is defined by

$$
\mathtt{R}^\mu{}_{\nu\rho\sigma} v^\nu = 2(\mathtt{D}_\rho \mathtt{D}_\sigma - \mathtt{D}_\sigma \mathtt{D}_\rho) v^\mu \ , \tag{4.22}
$$

for any (commuting) space-time vector v^μ. By multiplying by e^a_μ, we can relate it to the spin curvature

$$
\begin{aligned}
e^a_\mu \mathtt{R}^\mu{}_{\nu\rho\sigma} v^\nu &= 2(\mathtt{D}_\rho \mathtt{D}_\sigma - \mathtt{D}_\sigma \mathtt{D}_\rho) e^a_\mu v^\mu \\
&= 2(\mathcal{D}_\rho \mathcal{D}_\sigma - \mathcal{D}_\sigma \mathcal{D}_\rho) e^a_\mu v^\mu \\
&= \mathcal{E}_{bc}{}^a \mathcal{R}^b_{\rho\sigma} e^c_\nu v^\nu \ .
\end{aligned}
\tag{4.23}
$$

Thus $\mathtt{R}^\mu{}_{\nu\rho\sigma}$ is zero as a result of $\mathcal{R}^b$ being zero. Furthermore, if we replace (2.24) by the stronger conditions

$$
\begin{aligned}
\omega^a_\mu \mathcal{R}^b_{\rho\sigma} &= \mathcal{R}^b_{\rho\sigma} \omega^a_\mu \ , \\
e^a_\mu \mathcal{R}^b_{\rho\sigma} &= q^b \, \mathcal{R}^b_{\rho\sigma} e^a_\mu \ ,
\end{aligned}
\tag{4.24}
$$

then (4.23) implies that the space-time curvature commutes with everything.

5 Point Sources

Here we include point sources in the field equations in a standard manner[10], i.e. by including delta function contributions to the field equations. We presume that a particle traces out a world line $y^\mu(\tau)$ on M, τ being a real parameter. We then endow the particle with 'momentum' and 'angular momentum' degrees of freedom, $p^a(\tau)$ and $j^a(\tau)$, respectively. Now instead of (1.23), we have

$$\begin{aligned} \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}\mathcal{R}^a_{\nu\lambda}(x) &= \int d\tau\ \delta^3(x-y(\tau))p^a(\tau)\partial_\tau y^\mu\ , \\ \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}\mathcal{T}^a_{\nu\lambda}(x) &= \int d\tau\ \delta^3(x-y(\tau))j^a(\tau)\partial_\tau y^\mu\ , \end{aligned} \tag{5.1}$$

where $\partial_\tau = \frac{d}{d\tau}$, $\epsilon^{\mu\nu\lambda}$ is totally antisymmetric in the space-time indices, and we introduce the 'gravitational' coupling constant κ. Below we give some properties of the particle degrees of freedom $p^a(\tau)$ and $j^a(\tau)$.

From the commutation relations (2.25), $p^a(\tau)$ and $j^a(\tau)$ are not c-numbers. For consistency with (2.25) we require

$$\begin{aligned} [p^a,p^b] \;=\; [p^a,j^b]_{q^{-a}} \;=\; [j^a,j^b]_{q^{b-a}} &= 0\ , \\ [\omega^a,p^b] = [e^a,p^b]_{q^b} \;=\; [\omega^a,j^b]_{q^{-a}} \;=\; [e^a,j^b]_{q^{b-a}} &= 0\ . \end{aligned} \tag{5.2}$$

Thus the particle momentum commutes with itself, but not with the angular momentum. It then follows that in the absence of source terms for the torsion the Einstein-Cartan equations are the usual ones.

Under gauge transformations, p^a and j^a transform as $\mathcal{R}^a$ and $\mathcal{T}^a$ (2.19),

$$\begin{aligned} \delta_R p^c &= \mathcal{E}_{ab}{}^c p^a \tau^b\ , \\ \delta_R j^c &= \mathcal{E}_{ab}{}^c (p^a\rho^b - \tau^a j^b)\ , \end{aligned} \tag{5.3}$$

where here ρ^b and τ^a denote infinitesimal functions of the particle's space-time coordinates $y^\mu(\tau)$. From (2.21), we get the commutation relations between these functions and p^a and j^a,

$$\begin{array}{ll} [\tau^a,p^b] = 0\ , & [\tau^a,j^b]_{q^{-a}} = 0\ , \\ [\rho^a,p^b]_{q^b} = 0\ , & [\rho^a,j^b]_{q^{b-a}} = 0\ . \end{array} \tag{5.4}$$

By substituting (5.1) into the Bianchi identities (2.27), we get the usual equations of motion for $p^a(\tau)$ and $j^a(\tau)$,

$$\begin{aligned} \partial_\tau p^c &= \mathcal{E}_{ab}{}^c p^a \omega^b_\mu \partial_\tau y^\mu\ , \\ \partial_\tau j^c &= \mathcal{E}_{ab}{}^c (p^a e^b_\mu - \omega^a_\mu j^b)\partial_\tau y^\mu\ . \end{aligned} \tag{5.5}$$

From them we find the usual conserved quantities

$$\mathcal{C}_1 = p_a p^a \quad \text{and} \quad \mathcal{C}_2 = p_a j^a\ , \tag{5.6}$$

the first being analogous to the mass squared, and the second being analogous to the mass times spin. However here, although $\mathcal{C}_1$ is in the center of the algebra generated by p^a , j^a, e^a and ω^a, $\mathcal{C}_2$ is not, since

$$[\mathcal{C}_2, p^b]_{q^b} = [\mathcal{C}_2, j^b]_{q^b} = 0 \ . \tag{5.7}$$

We therefore conclude that particle sources have an exotic spin.

6 Particle Lagrangian

Here we write down an action for the above mentioned point sources. By extremizing the particle action along with $\mathcal{S}_{CS}$ we will recover both the field equations (5.1), as well as the particle equations (5.5). In addition, the action is consistent with the commutation properties (5.2).

Following [11],[12], we write the momentum and angular momentum variables in terms of group variables, the group now being $ISO_q(2,1)$. We therefore again utilize the matrix elements ℓ_{ab} and z_a, only here they are functions on the particle world line, $\ell_{ab} = \ell_{ab}(\tau)$, $z_a = z_a(\tau)$. We shall assume the commutational properties (1.2).

We now express p^a and j^a according to

$$p^a = \ell^{ba}\hat{t}_b \ , \quad j^a = \ell^{ba}\hat{s}_b + \mathcal{E}^{abc}p_b(\ell^{-1}z)_c \ , \tag{6.1}$$

where $\hat{t}_b$ and $\hat{s}_b$ are to be regarded as constants. In order for (6.1) to be consistent with the commutational properties (5.2) and (1.2), these constants cannot be c-numbers. Instead we can assume that all the commutation relations with $\hat{s}_b$ and $\hat{t}^b$ are trivial except for

$$\hat{s}^a \ell_c{}^b = q^b \ \ell_c{}^b \hat{s}^a \ . \tag{6.2}$$

The set of all p^a and j^a satisfying (6.1) defines the adjoint orbit of $ISO_q(2,1)$. Using (6.1) and (1.17), p^a and j^a gauge transform according to the adjoint action of $ISO_q(2,1)$, i.e. as in (5.3).* In other words, gauge variations of

$$p^a J_a + j^a P_a \tag{6.3}$$

are the same as for F in (2.18). Upon introducing the covariant derivative D_τ of ℓ and z, we can define another set of quantities which gauge transform in the same way as (6.3):

$$\frac{1}{2}\mathcal{E}^{dbc}(\ell^{-1}D_\tau\ell)_{bc}J_d + (\ell^{-1}D_\tau z)^d P_d \ , \tag{6.4}$$

where

$$\frac{1}{2}\mathcal{E}^{abc}(\ell^{-1}D_\tau\ell)_{bc} = \frac{1}{2}\mathcal{E}^{abc}(\ell^{-1}\partial_\tau\ell)_{bc} - \omega^a_\tau \ , \quad (\ell^{-1}D_\tau z)^a = (\ell^{-1}\partial_\tau z)^a - e^a_\tau \ , \tag{6.5}$$

*Gauge transformations here differ from those in reference [12]. For the latter they correspond to left transformations of $ISO(2,1)$, while here they correspond to right transformations of $ISO_q(2,1)$.

and ω^a_τ and e^a_τ are the connection one forms evaluated on the particle world-line, i.e. $\omega^a_\tau = \omega^a_\tau(\tau) = \omega^a_\mu(y(\tau))\partial_\tau y^\mu$ and $e^a_\tau = e^a_\tau(\tau) = e^a_\mu(y(\tau))\partial_\tau y^\mu$.

The scalar product $< , >$ defined in Section 3 is adjoint invariant. Hence, we can define a gauge invariant particle Lagrangian L_p as the scalar product of (6.3) and (6.4):

$$\begin{aligned} L_p &= < p^a J_a + j^a P_a \;,\; \frac{1}{2}\mathcal{E}^{dbc}(\ell^{-1}D_\tau\ell)_{bc}J_d + (\ell^{-1}D_\tau z)^d P_d > \\ &= p_a(\ell^{-1}D_\tau z)^a + \frac{1}{2}q^{-a}j_a\mathcal{E}^{abc}(\ell^{-1}D_\tau\ell)_{bc} \\ &= L_0 - p_a e^a_\tau - q^{-a}j_a\omega^a_\tau \;, \end{aligned} \tag{6.6}$$

where L_0 is the free particle Lagrangian. Using the identities (1.4) , (1.6) and (1.9), the latter can be expressed according to

$$L_0 = (z^a\hat{t}^b + \frac{1}{2}\mathcal{E}^{abc}\hat{s}_c)(\partial_\tau\ell\ell^{-1})_{ab} \;, \tag{6.7}$$

up to total derivative terms.

It is now straightforward to obtain the field equations (5.1) by extremizing the action

$$\frac{\kappa}{2}\mathcal{S}_{CS} + \int d\tau L_P \tag{6.8}$$

with respect to variations in the components of dreibein and spin connection one forms. Here we also need

$$\delta\omega^a \wedge \mathcal{T}^b = q^{-a}\;\mathcal{T}^b \wedge \delta\omega^a \;. \tag{6.9}$$

Getting the particle equations of motion (5.5) requires a little more work. Upon extremizing the particle action $\int d\tau L_P$ with respect to variations in the Lorentz vector z^a we get

$$\begin{aligned} \delta\int d\tau L_P &= \int d\tau \delta z^e \hat{t}^d \Big\{(\partial_\tau\ell\ell^{-1})_{ed} - \mathcal{E}_{abc}\ell_d{}^b\ell_e{}^c\omega^a_\tau\Big\} \\ &= -\int d\tau\delta z^e \ell_e{}^c(\partial_\tau p_c - \mathcal{E}_{abc}p^a\omega^b_\tau) \;, \end{aligned} \tag{6.10}$$

where we integrated by parts and used

$$\delta z^a \ell_c{}^b = q^b\;\ell_c{}^b\delta z^a \;. \tag{6.11}$$

The first equation in (5.5) immediately follows from setting this variation equal to zero. The remaining equation in (5.5) can be recovered by considering right variations of ℓ, as in (1.17). Then

$$\delta(\partial_\tau\ell\ell^{-1})_{ab} = -\mathcal{E}_{abc}\ell^c{}_d\partial_\tau\tau^d \;. \tag{6.12}$$

Using this and (5.3), variations of the particle action take the form

$$\begin{aligned} \delta\int d\tau L_P &= \int d\tau\Big\{q^{-c}j_c\partial_\tau\tau^c + \mathcal{E}_{abc}(q^{-b}\tau^c j^a\omega^b_\tau - p^b\tau^c e^a_\tau)\Big\} \\ &= -\int d\tau q^{-c}\Big\{\partial_\tau j_c - \mathcal{E}_{abc}(\omega^b_\tau j^a - p^b e^a_\tau)\Big\}\tau^c \;, \end{aligned} \tag{6.13}$$

where we integrated by parts and used $[j^b, \tau^a]_{q^a} = 0$ and $[j^b, \omega^a_\tau]_{q^a} = 0$. The second equation in (5.5) immediately follows from setting this variation equal to zero.

7 Inclusion of a Cosmological Constant

In [1] it is shown that a cosmological constant can be included in $2+1$ dimensional gravity in such a way that the theory is still described by the Chern-Simons action. Now, however, the gauge group is not $ISO(2,1)$, but either $SO(2,2)$ or $SO(3,1)$, depending on the sign of the cosmological constant, λ. The algebra of $ISO(2,1)$ is generalized to

$$[J_a, J_b] = \epsilon_{abc} J^c \ , \qquad [J_a, P_b] = \epsilon_{abc} P^c \ , \qquad [P_a, P_b] = -\lambda \epsilon_{abc} J^c \ . \tag{7.1}$$

If λ is positive, this can be seen to be the $so(3,1)$ algebra, while if λ is negative, (7.1) is the $so(2,2)$ algebra. To see this we can express the algebra in terms of generators J_a and $P'_a = \frac{1}{\sqrt{|\lambda|}} P_a$.

The purpose of this Section is to construct a Chern-Simons theory based on a quantum deSitter or anti-deSitter group. Such q-groups do exist, as real forms of $SO_q(4)$ [13], and they have an associated bicovariant calculus and nondegenerate scalar product. However, we find that their contraction to $ISO_q(2,1)$ ($\lambda \to 0$) is singular, unless we take the limit $q \to 1$ first. Thus the q-CS gravity with cosmological constant is not continuously connected to the q-CS gravity with zero cosmological constant, at least for the deformation we use.

The minimal, multiparametric deformation of $SO(2n)$ described in [13] becomes, for $2n = 4$, a 1-parameter deformation and is given in terms of six generators

$$\{J^+_{-1}, J^+_0, J^+_1, J^-_{-1}, J^-_0, J^-_+\} \equiv \{T_i\}, \quad i = -1, ..., 4 \ , \tag{7.2}$$

obeying the commutation relations

$$[J^\pm_a, J^\pm_b] = \mathcal{E}_{abc} J^{\pm c}, \qquad [J^+_a, J^-_b]_{q^{2ab}} = 0 \tag{7.3}$$

The Λ matrix appearing in (2.10) can be read directly from (7.3); we have

$$\Lambda_{a\ \ b+3}^{c+3\ d} = \delta^d_a \delta^c_b q^{2ab}, \qquad \Lambda_{a+3\ b}^{c\ \ d+3} = \delta^d_a \delta^c_b q^{-2ab}, \qquad \Lambda^{cd}_{ab} = \Lambda^{c+3\ d+3}_{a+3\ b+3} = \delta^d_a \delta^c_b \tag{7.4}$$

with all the other components equal to zero. As a real algebra, (7.3) can be seen to be the $so(2,2)$ algebra, realized as the direct sum of two copies of $so(2,1)$, when $q \to 1$ ($so(3,1)$ can be obtained by taking a complex combination of the generators). Also note that the q-structure constants are the same as the undeformed ones.

The algebra defined by (7.3) satisfies all the conditions for a minimal bicovariant calculus, i.e. eqs. (2.5)- (2.10). The bicovariant calculus can be defined on the quantum group $Fun_q(SO(2,1) \otimes SO(2,1))$ generated by the 3×3 matrices ℓ_+ and ℓ_-, both of which satisfy the defining relations for $SO(2,1)$. Their matrix elements $\ell_{\pm ab}$ satisfy trivial commutation relations, except for

$$\ell_-{}^{ab} \ell_+{}^{cd} = q^{-2bd} \ell_+{}^{cd} \ell_-{}^{ab} \ . \tag{7.5}$$

The differential calculus on $Fun_q(SO(2,1)\otimes SO(2,1))$, along with a consistent set of left and right actions, can be constructed in a straightforward manner. From them we get the left invariant one forms

$$A^c_\pm = \frac{1}{2}\mathcal{E}_{ab}{}^c(\ell^{-1}_\pm d\ell_\pm)^{ab} \ . \tag{7.6}$$

It is clear that the quantum group $ISO_q(2,1)$ defined in Section 1 cannot be obtained from $Fun_q(SO(2,1)\otimes SO(2,1))$ by something analogous to a group contraction. It also seems reasonable that a similar statement can be made for the corresponding q-Lie algebras, and thus that the corresponding deformed CS theories cannot be continuously connected.

We next perform a rotation from (J_+,J_-) basis to the (J,P') basis, which is needed if one has hopes of describing gravity; P' is the generator P in (7.1) (in the $q\to 1$ limit) rescaled by a factor of $1/\sqrt{|\lambda|}$. To be definite we assume from now on λ to be negative (hence the gauge algebra to be $so_q(2,2)$); we have then

$$J_a = J^+_a + J^-_a \ , \qquad P'_a = J^+_a - J^-_a \ . \tag{7.7}$$

For $q\neq 1$ this change of basis is not uniquely defined, as we can consider linear combinations of J^-, J^+ with coefficients depending on q. Among them, (7.7) is the simplest one. We denote by $V_i, i=-1,4$,

$$V_i = A^j_i T_j \ , \tag{7.8}$$

the generators $J_a = V_a$ and $P'_a = V_{a+3}$, where the matrix A can be read off from (7.7). The new generators obey new deformed commutation relations

$$V_iV_j - \tilde{\Lambda}^{kl}_{ij}V_kV_l = \tilde{C}^r_{ij}V_r \tag{7.9}$$

with a new braiding matrix, $\tilde{\Lambda}$ and structure constants $\tilde{C}^k_{ij}$, given respectively by

$$\tilde{\Lambda}^{kl}_{ij} = A^r_i A^s_j \Lambda^{mn}_{rs}(A^{-1})^k_m (A^{-1})^l_n \tag{7.10}$$

and

$$\tilde{C}^k_{ij} = A^r_i A^s_j C^m_{rs}(A^{-1})^k_m \ . \tag{7.11}$$

The new structure constants are easily seen to be independent from the deformation parameter. In fact, they are identical to the undeformed ones, appearing in (7.1). In particular, they are skewsymmetric in the lower indices and it can be checked that the skewsymmetry is compatible with the $\tilde{\Lambda}$–skewsymmetry (2.7), which is a characteristic of minimal deformations. On the other hand the braiding matrix becomes more complicated than in the old basis. The simple structure $\Lambda^{kl}_{ij}\propto\delta^k_i\delta^l_j$ is not preserved; we have instead

$$\tilde{\Lambda}^{kl}_{ij} = \delta^l_i\delta^k_j F(q) \pm (\delta^{l+3}_i\delta^k_j - \delta^l_i\delta^{k+3}_j)G(q) - \delta^{l+3}_i\delta^{k+3}_j H(q) \ , \quad \text{for} \quad i,j\neq 0,3 \tag{7.12}$$

$$\tilde{\Lambda}^{kl}_{ij} = \delta^l_i\delta^k_j \ , \quad \text{for} \quad i,j=0,3 \tag{7.13}$$

where $i+3$ is to be intended as $i+3$ mod 6; the plus sign in (7.12) holds for $i=j,j+3$. The functions of q appearing in (7.12) are respectively

$$F(q) = \frac{1}{4q^2}(q^2+1)^2 \ , \quad G(q) = \frac{1}{4q^2}(q^4-1) \ , \quad H(q) = \frac{1}{4q^2}(q^2-1)^2 \ . \tag{7.14}$$

Note that both G and H go to zero for $q \to 1$, while $F \to 1$, so that the deformed algebra (7.9) becomes the Lie algebra (7.1) up the factor λ. Restoring the factors of λ in the algebra (7.9) it can be checked that the limit $\lambda \to 0$ (no cosmological constant) is singular unless the limit $q \to 1$ is previously performed. Thus the deformed theory with cosmological constant is not continuously connected to the one with zero cosmological constant described in the previous Sections.

To write down Chern Simons gravity we need an invariant q–metric on the deformed algebra, according to (3.1). Since the structure constants are undeformed we discover that the metric itself is undeformed. There are two non–degenerate, invariant metrics on $so(4)$:

$$< J_a, J_b >= \eta_{ab} \quad , \qquad < P'_a, P'_b >= \eta_{ab} \quad , \qquad < J_a, P'_b >= 0 \quad , \tag{7.15}$$

$$< J_a, J_b >= 0 \quad , \qquad < P'_a, P'_b >= 0 \quad , \qquad < J_a, P'_b >=< P'_a, J_b >= \eta_{ab} \quad . \tag{7.16}$$

The metric which makes the (undeformed) CS Lagrangian equal to the Einstein-Cartan Lagrangian of gravity (with cosmological constant) is the latter, as pointed out in [1]. The former gives indeed a term which, added to the Lagrangian, doesn't change the equations of motion. Due to the metric being undeformed, the expression for the Lagrangian is the same as the undeformed one (the equations of motion were already known to be unchanged as shown in [5]). The deformation is hidden in the noncommutativity of the connection components. The CS Lagrangian (3.4) for the present q–group becomes:

$$\mathcal{L}_{CS} = [d\omega^c + \frac{1}{3}\mathcal{E}_{ab}{}^c(\omega^a \wedge \omega^b + e'^a \wedge e'^b)] \wedge e'_c \ + \ [de'^c + \frac{1}{3}\mathcal{E}_{ab}{}^c(\omega^a \wedge e'^b + e'^a \wedge \omega^b)] \wedge \omega_c \ , \tag{7.17}$$

where the primed connections are rescaled by a factor of $\sqrt{|\lambda|}$. Restoring the factors of λ and rescaling the lagrangian by and overall $\sqrt{|\lambda|}$ we get the undeformed Einstein–Cartan Lagrangian of $2+1$ gravity

$$\mathcal{L}_{CS} = (d\omega^c + \frac{1}{3}\mathcal{E}_{ab}{}^c\omega^a \wedge \omega^b) \wedge e_c + [de^c + \frac{1}{3}\mathcal{E}_{ab}{}^c(\omega^a \wedge e^b + e^a \wedge \omega^b)] \wedge \omega_c - \frac{\lambda}{3}\mathcal{E}_{ab}{}^c e^a \wedge e^b \wedge e_c \tag{7.18}$$

If we try to reorder the terms containing both the spin connection and the dreibein in a given order, we have to use (2.15) and (7.12), (7.13), which will introduce a q–dependence in the Lagrangian. The equations of motion are, as expected, undeformed, for a particular ordering of the connections:

$$\begin{aligned} \mathcal{R}^c &= -\frac{\lambda}{2}\mathcal{E}_{ab}{}^c e^a \wedge e^b \\ \mathcal{T}^c &= 0 \ , \end{aligned} \tag{7.19}$$

where $\mathcal{R}$ and $\mathcal{T}$ are now defined as

$$\begin{aligned} \mathcal{R}^c &= d\omega^c + \frac{1}{2}\mathcal{E}_{ab}{}^c\omega^a \wedge \omega^b \ , \\ \mathcal{T}^c &= de^c + \frac{1}{2}\mathcal{E}_{ab}{}^c(\omega^a \wedge e^b + e^a \wedge \omega^b) \ . \end{aligned} \tag{7.20}$$

Such a theory is invariant under gauge transformations (2.1), which can be explicitly written as

$$\begin{aligned}\delta e^c &= d\rho^c + \mathcal{E}_{ab}{}^c(\omega^a\rho^b + e^a\tau^b) &(7.21)\\ \delta\omega^c &= d\tau^c + \mathcal{E}_{ab}{}^c(\omega^a\tau^b - \lambda e^a\rho^b)\ . &(7.22)\end{aligned}$$

Thus, a deformed theory of Einstein–Cartan gravity with cosmological constant exists with many of the features of the undeformed one.

Unlike what happens in Section 4 for zero cosmological constant, here we are not able to give a metric formulation of the theory. That is to say, it is not possible to construct a metric tensor out of the dreibeins, which is symmetric, and invariant under local Lorentz transformations. Also, it is not meaningful for the theory described above to have pure Lorentz transformations, as the non–commutativity of the gauge parameters with the connection components (2.3) and the structure of the braiding matrix generate terms containing the Lorentz parameter out of terms which do not contain it. To illustrate these things let us perform a gauge transformation of a bilinear in the dreibeins. We have

$$\delta(e^a\wedge e^b) = [d\rho^a + \mathcal{E}_{cd}{}^a(\omega^c\rho^d + e^c\tau^d)]\wedge e^b + e^a\wedge[d\rho^b + \mathcal{E}_{cd}{}^b(\omega^c\rho^d + e^c\tau^d)]; \qquad (7.23)$$

to compare the two terms we have to put all the gauge parameters on one side, for example on the right. Consider the first term of the sum and commute ρ^d with e^b; we get

$$\delta(e^a\wedge e^b) = [d\rho^a\wedge e^b - \mathcal{E}_{cd}{}^a(-\frac{H(q)}{\lambda}\omega^c\wedge\omega^b\tau^d + ...] + e^a\wedge[d\rho^b + \mathcal{E}_{cd}{}^b(\omega^c\rho^d + e^c\tau^d)]; \quad (7.24)$$

we have obtained a factor containing the spin connection and the gauge parameter τ, which cannot be cancelled by another factor. Also, the same term shows that it is not possible to perform pure Lorentz transformations. Hence we conclude that a metric tensor cannot be constructed for this theory, though the Lagrangian which we have exhibited is a legitimate deformation of the Einstein–Cartan Lagrangian of gravity with cosmological constant.

8 Concluding Remarks.

The increasing mathematical interest in quantum Lie groups, quantum Lie algebras and deformed affine Lie algebras [15], as well as the rôle played by them in such physical problems as $1+1$ solvable models, has motivated the construction of $q-$deformed gauge field theories. To this end, the construction of bicovariant calculi on quantum groups has been extensively studied leading to the formulation of consistent Yang-Mills theories for different quantum groups, including the $q-$deformed unitary groups. [16] A further development, utilized in this article, has been the construction of deformed CS theory. This was applicable for minimally deformed groups whose associated quantum Lie algebras are endowed with a nondegenerate invariant scalar product.

Despite its elegance and formal consistency, the physical relevance of a *classical* field theory formulated in terms of abstract noncommuting variables needs to be clarified. The results obtained in this paper may provide a first step in that direction. Instead of constructing new *exotic* theories with such variables, one can search for *hidden* structures in well known field theories. (An analogous result was found within the framework of the classical mechanics of a rigid rotor [18].) In the present paper we have done this by showing that (torsion-free) Einstein general relativity may be reformulated not only as a Poincaré group gauge theory with pure CS action, but more generally, as a $q-$CS theory based on a quantum Poincaré gauge group.

¿From our theory, one has the possibility of gaining new insights for gravity, apart from the possibility of coupling it to matter endowed with exotic properties such as those described in Section 6. First, one may notice that the theory is equipped with a different canonical formalism for each value of q, with the standard one recovered when $q \to 1$. Now if one were able to quantize the theory, perhaps by extending functional integration to the case of noncommuting variables taking value in a quantum Lie algebra, it might be possible to generalize known mathematical and physical results of CS theory. For example, by generalizing the results of E. Witten [14], one could associate new invariants to a class of knots by taking the expectation value of path-ordered exponentials of line integrals along closed space-time curves. The line integrals would be functionals of quantum Lie algebra valued gauge fields. More generally, a quantum version of a $q-$deformed field theory involving two parameters (the deformation dimensionless quantity q and Planck's constant) would be instructive. One could imagine a scenario where q ends up playing the rôle of a regularization parameter. Finally, the most immediate and physically relevant follow up of our work is the search for *hidden* deformed structures for $3+1$ dimensions Einstein general relativity. As we have already mentioned, this search has been successful, and we will report on it in a forthcoming article [4].

REFERENCES

[1] E. Witten, *Nucl. Phys.* **B311** (1988) 46.

[2] A. Achúcarro and P.K. Townsend, *Phys. Lett.* **B180** (1986) 89.

[3] P. Salomonson, B.-S. Skagerstam and A. Stern, *Nucl. Phys.* **B347** (1990) 769; C. Vaz, "Exact Solutions in Topological Gravity", UCEH-102-92; in Conf. on Topics in Quantum Gravity, Cincinnati 1992, eds F. Mansouri and J.J. Scanio: Quantum Gravity and Beyond; S. Carlip, *Canadian Gen. Rel.* (1993) 215; G. Bimonte, K. S. Gupta and A. Stern, *Int. J. Mod. Phys.* **A8** (1993) 653. M. Hayashi, *Nucl. Phys.* **B405** (1993) 228; M. Hayashi, F. P. Zen, *Prof. Theor. Phys.* **91** (1994) 361. W. G. Unruh and P. Newbury, *Int. J. Mod. Phys.* **D3** (1994) 131; K. Ezawa, *Phys. Rev.* **D49** (1994) 5211; A. P. Balachandran, L. Chandar and A. Momen, "Edge States in Canonical Gravity", SU-4240-610, talk given at 17th annual MRST meeting, gr-qc 9506006; *Int. J. Mod. Phys.* **A12** (1997) 625.

[4] G. Bimonte, R. Musto, A. Stern and P. Vitale, in preparation.

[5] G. Bimonte, R. Musto, A. Stern and P. Vitale, "Deformed Chern-Simons Theories", Alabama preprint UAHEP 975 and Napoli preprint DSF 16/97, hep-th/9704118, Phys. Lett **B**, in press.

[6] S. L. Woronowicz, *Comm. Math. Phys.* **111** (1987) 613; *ibid.* **122** (1989) 125. D. Bernard, *Prog. Theor. Phys. Suppl* **102** (1990) 49; D. Bernard and A. Le Clair, *Comm. Math. Phys.* **142** (1991) 99. B. Zumino, *Proc. Math. Phys X Leipzig* Springer–Verlag (1992); P. Schupp, P. Watts and B. Zumino, *Comm. Math. Phys.* **157** (1993) 305. S. Watamura, *Comm. Math. Phys.* **158** (1993) 67. P. Aschieri and L. Castellani, *Int. J. Mod. Phys.* **A8** (1993) 1667.

[7] L. Castellani, *Mod. Phys. Lett.* **A9** (1994) 2835; *Phys. Lett.* **B292** (1992) 93; *Phys. Lett.* **B327** (1994) 22.

[8] L. Castellani, *Commun. Math. Phys.* **171** (1995) 383.

[9] See for example, R. Jackiw, *Phys. Rev. Lett.* **41** (1978) 1635.

[10] See for example, K. Gupta and A. Stern, *Phys Rev.* **D44** (1991) 2432.

[11] A. P. Balachandran, G. Marmo, B. S. Skagerstan and A. Stern, *Phys. Lett.* **B89** (1980) 199; "Gauge Symmetries and Fibre Bundles, Applications to particle Dynamics", Lecture Notes in Physics **188**, Springer–verlag, Berlin, 1982.

[12] B. S. Skagerstam and A. Stern, *Int. J. Mod. Phys.* **A5** (1990) 1575.

[13] P. Aschieri and L. Castellani, *Int. J. Mod. Phys.* **A11** (1996) 4513.

[14] E. Witten, *Commun. Math. Phys.* **121** (1989) 351.

[15] M. Jimbo, *Lett. Math. Phys.* **10** (1985) 63. V. G. Drinfeld, *Quantum Groups* ICM Proc. New York, Berkeley 1986. L. D. Faddeev, N. Yu. Reshetikin and L. A. Takhtajan,*Algebra and Analysis* **1** (1987) 178. S. Majid, *Int. J. Mod. Phys.* **A5** (1990) 1.

[16] J. M. Maillet and F. Nijoff *Phys. Lett.* **B229** (1989) 71. I. Ya. Arefeva and I. V. Volovich, *Mod. Phys. Lett* **A6** (1991) 893. M. Hirayama *Prog. Theor. Phys.* **88** (1992) 111. A. P. Isaev and Z. Popowicz, *Phys. Lett.* **B281** (1992) 271; *ibid.* **B307** (1993) 353. T. Brezinski and S. Majid, *Phys. Lett* **B 298** (1993) 339; *Comm. Math. Phys.* **157** (1995) 591; *ibid.* **167** (1995) 235; P. M. Hajac, "Strong Connections and $U_q(2)$ YM Theory on Quantum Principal Bundles" hep-th/9406129. M. Durdevic, "Quantum Principal Bundles and Corresponding Gauge Theories q-alg/9507021;

[17] A. Sudbery, *Phys. Lett.* **B375** (1996) 75. P. Watts, "Toward a q–Deformed Standard Model" Preprint hep-th/9603143.

[18] G. Marmo, A. Simoni and A. Stern, *Int. J. Mod. Phys.* **A10** (1995) 99.